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The Veldkamp space of multiple qubits

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Abstract
We introduce a point-line incidence geometry in which the commutation relations of the real Pauli group of multiple qubits are fully encoded. Its points are pairs of Pauli operators differing in sign, and each line contains three pairwise commuting operators any of which is the product of the other two (up to sign). We study the properties of its Veldkamp space enabling us to identify subsets of operators which are distinguished from the geometric point of view. These are geometric hyperplanes and pairwise intersections. Among the geometric hyperplanes, one can find the set of self-dual operators with respect to the Wootters spin-flip operation well known from studies concerning multiqubit entanglement measures. In the two- and three-qubit cases, a class of hyperplanes gives rise to Mermin squares and other generalized quadrangles. In the three-qubit case, the hyperplane with points corresponding to the 27 Wootters self-dual operators is just the underlying geometry of the $E_{6(6)}$ symmetric entropy formula describing black holes and strings in five dimensions.

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1. Introduction

The importance of generalized Pauli groups in the study of quantum systems with finite-dimensional Hilbert spaces is well known. The main application of this group within the field of quantum information is related to quantum error correcting codes [1]. The construction of such codes is naturally facilitated within the so-called stabilizer formalism [1–3]. Here it is recognized that the basic properties of error correcting codes are related to the fact that two operators in the Pauli group are either commuting or anticommuting. This property is encoded into the structure of an Abelian group (the central quotient of the Pauli group), with a natural symplectic structure. As a next step it was later realized that this commutation algebra for a multiqubit system is encoded in the totally isotropic subspaces of this underlying symplectic vector space, that is, a symplectic polar space of order 2 [4].

Finite geometric concepts in connection with multiqubit Pauli groups also arose in different contexts, e.g. in connection with discrete phase spaces [5], topological quantum computation [6] and notably in the so-called black hole analogy [7, 8]. In the latter context, it was shown that there is a mathematical connection between the Bekenstein–Hawking entropy formula of black holes and black strings and certain finite geometric objects related to the three-qubit real Pauli group. (For a review of the black hole analogy, see the paper of Borsten *et al* [9] and references therein.) More precisely, in a previous paper [8] an explicit connection has been established between the structure of one type of the geometric hyperplanes of the split Cayley hexagon of order 2 based on the Pauli group for three qubits and the entropy formula for five-dimensional black hole and string solutions well known to string theorists. Apart from their use in string theory these studies also emphasized an important connection between the structure of incidence geometries and their finite automorphism groups realized in terms of quantum gates of quantum information theory [7, 8]. These spirit groups, such as the Weyl groups $W(E_6)$ and $W(E_7)$ and the simple group $PSL(2, 7)$ as finite subgroups of the infinite discrete U-duality group known from string theory, have been linked to the Clifford group of quantum computation [7, 8, 10].

The aim of this paper is twofold. Firstly, we would like to draw the attention to certain subsets of the n-qubit-generalized Pauli group which are distinguished from the finite geometric point of view. These are points and lines of the Veldkamp space of an incidence geometry naturally associated with the real Pauli group of n qubits. Some of them have a clear quantum information theoretic meaning in terms of the Pauli operators, but for the others this meaning is yet to be found. Secondly, since these subsets as geometric hyperplanes of our incidence geometry are arising quite naturally also in the black hole analogy, we would like to provide a rigorous mathematical frame for these interesting constructions. Such considerations might possibly pave the way for a deeper understanding of this fascinating topic.

The organization of the paper is as follows. In section 2, we fix our conventions concerning the incidence geometry of the Pauli group. Here we introduce the important notion of a geometric hyperplane. In section 3, besides providing the basic properties of our incidence geometry, we prove that in this geometry no geometric hyperplane is contained in the other. We also show that a pair of geometric hyperplanes naturally gives rise to a third one. These considerations lead us, in section 4, to initiate a detailed study of the structure of the Veldkamp space, another incidence geometry associated with our initial one, with its points being the geometric hyperplanes. Here we provide an algebraic characterization for the Veldkamp points and establish different relationships between them. In section 5, we study the orbits of the action of the symplectic group on the hyperplanes, with the result that there are five types of Veldkamp lines. In section 6 by investigating the intersection properties of these lines, we obtain a full classification for them. To put these abstract considerations into physical context, we will examine some important and special cases in section 7. The conclusions are left for section 8.

2. The incidence geometry of the Pauli group

First we briefly summarize the relevant definitions from finite geometry. These can be found in e.g. [11]. The basic object we will be working with is the incidence structure whose definition is given here.

Definition. *The triple (P, L, I) is called an incidence structure (or point-line incidence geometry) if P and L are disjoint sets and $I \subseteq P \times L$ is a relation. The elements of P and L are called points and lines, respectively. We say that $p \in P$ is incident with $l \in L$ if $(p, l) \in I$.*

The concept of an incidence structure is quite general. We may restrict ourselves to those that are called simple, having the property that no two lines are incident with exactly the same points. In simple incidence structures the lines may be identified with the sets of points they are incident with, so we can think of these as a set P together with a subset $L \subseteq 2^P$ of the power set of P. Then $(P, L, \in)$ is an incidence structure in the usual sense. In what follows we will do this identification, i.e. the points incident with a line will be called the elements of that line.

In a point-line geometry, there are distinguished sets of points called geometric hyperplanes [12].

Definition. *Let (P, L, I) be an incidence structure. A subset $H \subseteq P$ of P is called a geometric hyperplane if the following two conditions hold:*

(H1) $(\forall l \in L) : (|H \cap l| = 1 \text{ or } l \subseteq H)$

(H2) $H \neq P$.

Clearly, the subsets H satisfying only (H1) are exactly the geometric hyperplanes and the set P of all points.

We will associate a point-line incidence geometry with the generalized real Pauli group of n qubits for all n. This group can be constructed in the following way. Let us define the 2×2 matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

Observe that these matrices satisfy $X^2 = Z^2 = I$, where I is the 2×2 identity matrix. The product of the two will be denoted by $Y = ZX = -XZ$. The n-qubit real Pauli group is the subgroup of $GL(2^n, \mathbb{R})$ consisting of the n-fold tensor (Kronecker) products of these four matrices and their negatives. The shorthand notation $AB \ldots C$ will be used for the tensor product $A \otimes B \otimes \cdots \otimes C$ of one-qubit Pauli group elements $A, B, \ldots, C$, i.e. we will omit the tensor product sign $\otimes$. The center of this group is the same as its commutator subgroup and contains only the identity element and its negative ($II \ldots I$ and $-II \ldots I$).

It was shown [3, 13] that the central quotient of the Pauli group has the structure of a symplectic vector space over the field with two elements. The dimension of this vector space is $2n$, and as the center of the Pauli group contains only the identity matrix and its negative, the vector addition corresponds to matrix multiplication up to sign. The symplectic form is induced by the commutator and has value 0 if (arbitrary preimages of) the two arguments commute and a value of 1 if they anticommute.

Elements of this vector space will be denoted by their representatives in the Pauli group; for the zero vector we will simply use 0. Using the ordered basis consisting of elements containing exactly one Z or X and no Ys, we will identify this space with $\mathbb{Z}_2^{2n}$ in the following way:

$$\begin{aligned} I \ldots IIX &\leftrightarrow (0, 0, 0, 0, \ldots, 0, 1) \\ I \ldots IIZ &\leftrightarrow (0, 0, 0, 0, \ldots, 1, 0) \\ &\vdots \\ XII \ldots I &\leftrightarrow (0, 1, 0, 0, \ldots, 0, 0) \\ ZII \ldots I &\leftrightarrow (1, 0, 0, 0, \ldots, 0, 0). \end{aligned} \tag{2}$$

In this basis, the matrix of the symplectic form is of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3}$$

Note that we are working in characteristic (2); therefore, every alternating matrix is also symmetric. The symplectic form and the symplectic vector space $\left(\mathbb{Z}_2^{2n}, \langle\cdot,\cdot\rangle\right)$ will be denoted by $\langle\cdot,\cdot\rangle$ and V_n, respectively.

Recall that the projective space $PG(2n-1, 2)$ consists of the nonzero subspaces of the $2n$-dimensional vector space over the two-element field $\mathbb{Z}_2$. The points of the projective space are one-dimensional subspaces of the vector space, and more generally, k-dimensional subspaces of the vector space are $(k-1)$-dimensional subspaces of the corresponding projective space.

A subspace of a symplectic vector space (and also the subspace in the corresponding projective space) is called isotropic if there is a vector in it which is orthogonal to the whole subspace, and totally isotropic if the subspace is orthogonal to itself. In the case of one- and two-dimensional (linear) subspaces, the two notions coincide.

Our incidence geometry consists of the one- and two-dimensional isotropic subspaces, i.e. the points and isotropic lines of the projective space $PG(2n-1, 2)$. The collinearity graph of this geometry was studied in [14]. Since the multiplicative group of the invertible elements in the two-element field is trivial, the points of this projective space can be identified with nonzero vectors. For the later reference, we present here the precise definition using our conventions.

Definition. *Let $n \in \mathbb{N}+1$ be a positive integer and V_n be the symplectic $\mathbb{Z}_2$-linear space as above. The incidence structure $\mathcal{G}_n$ of the n-qubit-generalized Pauli group is $(P, L, \in)$, where $P = V_n \setminus \{0\}$,*

$$L = \{\{a, b, a+b\} | a, b \in P, a \neq b, \langle a, b\rangle = 0\}, \tag{4}$$

and $\in$ is the set theoretic membership relation.

In the language of Pauli operators, we can say that points of this incidence geometry are the pairs of generalized Pauli operators differing only in a factor of -1 except for the identity element and its negative. On every line, there are three points which are represented by three pairwise commuting operators any two of which have the third as their product (up to sign).

Our aim will be to find the geometric hyperplanes of the above defined incidence geometry and to interpret them as special subsets of the real generalized Pauli group.

3. Basic properties

Firstly, we calculate the cardinalities of the point, line sets and the number of lines incident with one point. Since points in $\mathcal{G}_n = (P, L, \in)$ are identified with nonzero vectors of V_n, it follows that

$$|P| = 2^{2n} - 1 = 4^n - 1. \tag{5}$$

The points collinear with a given point x are

$$C_x = \{p \in P | \langle x, p\rangle = 0\} = \{p \in V_n | \langle x, p\rangle = 0\} \setminus \{0\}. \tag{6}$$

In other words, C_x is precisely the symplectic complement of the subspace spanned by x minus the zero vector. Hence, $|C_x| = 2^{2n-1} - 1$. Apart from $x \in C_x$, every element determines a line in $\mathcal{G}_n$ passing through x, and every such line is represented by two elements of C_x. It follows that the number of lines incident with a given point is $2^{2n-2} - 1 = 4^{n-1} - 1$. The total number of lines is the product of $|P|$ and the latter number divided by the number of points on a line:

$$|L| = \frac{(4^n - 1)(4^{n-1} - 1)}{3}. \tag{7}$$

As the next step, we derive some general properties of geometric hyperplanes of $\mathcal{G}_n$. In what follows, we exclude $\mathcal{G}_1$ from our consideration in some of the propositions as it is a degenerate case containing no lines at all. We have the following lower bound on the cardinality of a geometric hyperplane.

Lemma 1. *Let $n \in \mathbb{N} + 2$, $\mathcal{G}_n = (P, L, \in)$ and $H \subseteq P$ satisfying (H1). Then the inequality*

$$\frac{|P|}{3} \leqslant |H| \tag{8}$$

holds.

Proof. A subset $H \subseteq P$ satisfying (H1) must contain at least one point of every line. Since one point is incident with $4^{n-1} - 1$ lines, $|H|$ points can contain points from at most $|H|(4^{n-1} - 1)$ lines. Comparing this with the total number of lines, one obtains

$$|H|(4^{n-1} - 1) \geqslant |L| = \frac{(4^n - 1)(4^{n-1} - 1)}{3} = \frac{|P|(4^{n-1} - 1)}{3} \tag{9}$$

which implies the statement since $4^{n-1} - 1 > 0$ for $n \geqslant 2$. □

Remark. For $n = 1$ the statement does not hold: the empty set is a geometric hyperplane in $\mathcal{G}_1$.

Denoting the number of lines intersecting H in 1 point by N_1 and the number of lines fully contained in H by N_2 one can write

$$|H| \cdot (4^{n-1} - 1) = N_1 + 3N_2 \tag{10}$$

and obviously $N_1 + N_2 = |L|$. Solving this system of equations one obtains the formula

$$N_2 = \frac{1}{2}(4^{n-1} - 1)\left(|H| - \frac{|P|}{3}\right) \tag{11}$$

for the number of lines fully contained in a hyperplane. Since this must be non-negative, this also yields an alternative proof for the lemma above.

We now give a lower bound on the difference of cardinalities of two subsets of P satisfying (H1) such that one is contained in the other.

Lemma 2. *Let $n \in \mathbb{N} + 2$, $\mathcal{G}_n = (P, L, \in)$ and suppose that $A \subset B \subseteq P$ are two subsets of P satisfying (H1). Then*

$$\frac{3}{8}4^n \leqslant |B \setminus A|. \tag{12}$$

Proof. For all $p \in P$, let $N_p = C_p \setminus \{p\}$ denote the set of points collinear with but not equal to p, and for a collinear pair of points p and q let $N_{pq} = C_p \cap C_q \setminus \{p, q, p + q\}$ be the set of points collinear with all of the points of the line containing p and q minus the points of the line itself. Straightforward calculation shows that $|N_p| = 2^{2n-1} - 2$, $|N_x \cap N_y| = 2^{2n-2} - 3$ and $|N_{pq}| = 2^{2n-2} - 4$.

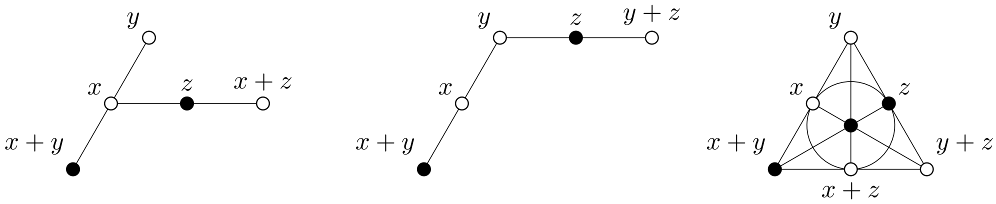

Figure 1. The three possible configurations of a point z incident with at least one of two given points x and y.

Let x be an element of the difference set $B \setminus A$, and pick a line passing through x. Since A and B both satisfy (H1) and x is contained in B but not in A, exactly one of the two other points on this line is in A and the third point y is contained in B.

Similar reasoning holds for every line passing through x or y, i.e. of each of these lines exactly one point is contained in A, and the whole lines are contained in B. For a point $z \in A$ on one of these lines but not on the line $\{x, y, x + y\}$, there are three different possibilities. Either z is collinear with precisely one of x or y or it is collinear with both. These cases are illustrated in figure 1 (filled and empty circles correspond to points of A and $B \setminus A$, respectively).

In the first two cases, B must also contain $x + z$ or $y + z$, respectively. There are

$$\frac{(|N_x \setminus N_y| - 1)}{2} + \frac{(|N_y \setminus N_x| - 1)}{2} \tag{13}$$

such points, and they correspond to the same number of points in $B \setminus A$.

In the latter case, when z is collinear with both x and y, it is also collinear with $x + y$, and this implies that the points $\{x, y, z, x + y, x + z, y + z, x + y + z\}$ together with the lines which are subsets of this point set form a Fano plane. Then the line $\{x + y, z, x + y + z\}$ is in A, and the points $x + z$ and $y + z$ are outside A. Since one such Fano plane contains four points outside the line $\{x, y, x + y\}$, it follows that the number of them is

$$\frac{|N_{xy}|}{4}. \tag{14}$$

Each Fano plane containing $\{x, y, x + y\}$ gives rise to two points in $B \setminus A$, which means that

$$\begin{aligned} |B \setminus A| &\geqslant 2 + \frac{(|N_x \setminus N_y| - 1)}{2} + \frac{(|N_y \setminus N_x| - 1)}{2} + 2 \cdot \frac{|N_{xy}|}{4} \\ &= 2 + (2^{2n-1} - 2) - (2^{2n-2} - 3) - 1 + \frac{2^{2n-2} - 4}{2} \\ &= \frac{3}{8} 4^n. \end{aligned} \tag{15}$$

□

Remark. Again, $n \geqslant 2$ is needed. In $\mathcal{G}_1$, all proper subsets of P are geometric hyperplanes allowing the difference to consist of a single element, but $\frac{3}{8}4^1 = \frac{3}{2} > 1$.

Now we are ready to prove an important fact about the geometric hyperplanes of $\mathcal{G}_n$ ($n \geqslant 2$).

Theorem 1. *Let $n \in \mathbb{N} + 2$, $\mathcal{G}_n = (P, L, \in)$ and suppose that $A, B \subset P$ are two geometric hyperplanes. Then $A \subseteq B$ implies $A = B$, i.e. no geometric hyperplane is contained in another one.*

Proof. Suppose that $A \subset B$ are geometric hyperplanes, one of which is a proper subset of the other. Then by lemma 1, we have that

$$\frac{4^n - 1}{3} \leqslant |A|. \tag{16}$$

But since $A \subset B \subseteq P$ and $B \subset P \subseteq P$ are two pairs of sets satisfying the conditions of lemma 2, we also have

$$\begin{aligned} |P| &= |A| + |B \setminus A| + |P \setminus B| \\ &\geqslant \frac{4^n - 1}{3} + 2 \cdot \frac{3}{8} 4^n \\ &= \frac{13}{12} 4^n - \frac{1}{3}, \end{aligned} \tag{17}$$

which contradicts equation (5). □

In our incidence geometry every line contains three points. This implies that a pair of geometric hyperplanes (A, B) gives rise to a third one, the complement of their symmetric difference which will be denoted by $A \boxplus B$:

Lemma 3. *Suppose that $A \neq B$ are geometric hyperplanes in $\mathcal{G}_n = (P, L, \in)$ where $n \geqslant 1$. Then the set*

$$A \boxplus B := \overline{A \triangle B} = (A \cap B) \cup (\overline{A} \cap \overline{B}) = \overline{\overline{A} \triangle \overline{B}} \tag{18}$$

(where $\triangle$ denotes the symmetric difference and $\overline{\cdot} = P \setminus \cdot$ is the complement) is also a geometric hyperplane.

Proof. Since $A \neq B$, the complement of the symmetric difference is not P.

We have to show that given a line $l \in L$, the set $l \cap (A \boxplus B)$ has an odd number of elements if $l \cap A$ and $l \cap B$ do so:

$$\begin{aligned} |l \cap (A \boxplus B)| &= |l \cap A \cap B| + |l \cap (P \setminus (A \cup B))| \\ &= |l \cap A \cap B| + |l \setminus (A \cup B)|. \end{aligned} \tag{19}$$

If any of A and B contains l then the latter term is 0, and the first term equals $|l \cap A|$ or $|l \cap B|$ both of which are odd.

If both of A and B meet l in a single point then either these points coincide or they are different points of l. In the first case, the first term is equal to 1 and the second equals 2, and in the second case the first term equals 0 and the second is $|l| - 2 = 1$. □

A simple observation about this operation is that

$$\begin{aligned} A \cap (A \boxplus B) &= A \cap ((A \cap B) \cup (\overline{A} \cap \overline{B}) \\ &= A \cap B \end{aligned} \tag{20}$$

and similarly for $B \cap (A \boxplus B)$. Moreover, any two of the triple $\{A, B, A \boxplus B\}$ determine the third one, since

$$\begin{aligned} A \boxplus (A \boxplus B) &= \overline{A \triangle \overline{A \triangle B}} \\ &= \overline{\overline{A} \triangle (\overline{A} \triangle \overline{B})} \\ &= B. \end{aligned} \tag{21}$$

In fact, $\boxplus$ makes the set of subsets of P satisfying (H1) a $\mathbb{Z}_2$-linear space with the set P as the zero vector.

4. The Veldkamp space

With certain incidence geometries one can associate another incidence geometry called its Veldkamp space whose points are geometric hyperplanes [12].

Definition. *Let $\Gamma = (P, L, I)$ be a point-line geometry. We say that Γ has Veldkamp points and Veldkamp lines if it satisfies the following conditions.*

(V1) For any hyperplane A, it is not properly contained in any other hyperplane B.
(V2) For any three distinct hyperplanes A, B and C, $A \cap B \subseteq C$ implies $A \cap B = A \cap C$.

If Γ has Veldkamp points and Veldkamp lines, then we can form the Veldkamp space $V(\Gamma) = (P_V, L_V, \supseteq)$ of Γ, where P_V is the set of geometric hyperplanes of Γ, and L_V is the set of intersections of pairs of distinct hyperplanes.

By theorem 1, the incidence geometry $\mathcal{G}_n$ has Veldkamp points for $n \geqslant 2$. Now we give the explicit form of geometric hyperplanes. To this end, let us introduce a quadratic form over V_n whose linearization is the symplectic form $\langle\cdot,\cdot\rangle$:

$$Q_0(x) = \sum_{i=1}^{n} a_i b_i, \tag{22}$$

where $x = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n) \in V_n$. It is easy to check that

$$Q_0(x+y) + Q_0(x) + Q_0(y) = \langle x, y\rangle. \tag{23}$$

It is also true that $Q_0(x) = 0$ iff the Pauli operators representing x are symmetric matrices. With every element p in V_n, we can associate a nondegenerate quadratic form

$$Q_p(x) = Q_0(x) + \langle p, x\rangle, \tag{24}$$

whose linearized form is the same as that of Q_0:

$$\begin{aligned} Q_p(x) + Q_p(y) + Q_p(x+y) &= Q_0(x) + \langle p, x\rangle + Q_0(y) + \langle p, y\rangle \\ &\quad + Q_0(x+y) + \langle p, x+y\rangle \\ &= Q_0(x) + Q_0(y) + Q_0(x+y) \\ &\quad + \langle p, x\rangle + \langle p, y\rangle + \langle p, x+y\rangle \\ &= \langle x, y\rangle. \end{aligned} \tag{25}$$

We will use these quadratic forms to characterize points of geometric hyperplanes.

An important concept is the Arf invariant of a quadratic form over a $\mathbb{Z}_2$-linear space which is the element of $\mathbb{Z}_2$ that occurs most often among the values of the form. It is not difficult to check that the Arf invariant of Q_x is $Q_0(x)$, where Q_0 is a quadratic form with the Arf invariant 0.

Lemma 4. *Let $n \in \mathbb{N} + 1$ be a positive integer, $\mathcal{G}_n = (P, L, \in)$ and $p \in V_n$ be any vector. Then the sets*

$$C_p = \{x \in P | \langle p, x\rangle = 0\} \tag{26}$$

and

$$H_p = \{x \in P | Q_p(x) = 0\} \tag{27}$$

satisfy (H1).

Proof. Since C_p is a projective subspace of co-dimension 1 (or 0 in the case of $x = 0$) in $PG(2n-1, 2)$ it intersects every line in this projective space (not only the isotropic ones) in either one or three points.

Now let $l = \{a, b, a+b\} \in L$ be a line in $\mathcal{G}_n$. Since it has three points, we only have to show that $\overline{H_p}$ intersects l in an even number of points. This is implied by

$$Q_p(a) + Q_p(b) + Q_p(a+b) = \langle a, b\rangle = 0. \tag{28}$$

□

Clearly, $C_0 = P$ but all other sets appearing in lemma 4 are geometric hyperplanes. In fact, the converse is also true, i.e. all geometric hyperplanes arise in this form.

Theorem 2. *Let $n \in \mathbb{N} + 1$, $\mathcal{G}_n = (P, L, \in)$, and $H \in P$ a subset satisfying (H1). Then either $H = C_p$ or $H = H_p$ for some $p \in V_n$.*

Proof. We prove by induction. For $n = 1$ one can check that the $8 = 2 \times 2^2$ possible subsets of P are indeed of this form. For $n \geqslant 2$ we can write n as the sum of two positive integers a and b. Then $V_n \simeq V_a \oplus V_b$ and the points of $\mathcal{G}_n$ are

$$\begin{aligned} P = {} & \{p_a \oplus 0 | p_a \in V_a \setminus \{0\}\} \cup \{0 \oplus p_b | p_b \in V_b \setminus \{0\}\} \\ & \cup \{p_a \oplus p_b | p_a \in V_a \setminus \{0\}, p_b \in V_b \setminus \{0\}\}. \end{aligned} \tag{29}$$

The first set in the union will be denoted by P_a, the second one by P_b, and the last one can naturally be identified with $P_a \times P_b$. The first two sets can be regarded as the point sets of $\mathcal{G}_a$ and $\mathcal{G}_b$, respectively. The latter two incidence structures arise as these points and the lines contained in the appropriate point sets.

Now let $H^{(a)} = H \cap P_a$ and $H^{(b)} = H \cap P_b$. Clearly, they satisfy (H1) in $\mathcal{G}_a$ and $\mathcal{G}_b$, so by the induction hypothesis they are either of the form C_{p_a} (C_{p_b}) or H_{p_a} (H_{p_b}) for some $p_a \in V_a$ ($p_b \in V_b$). In any case, since every point in P_a is connected with every point in P_b, the points in the intersections uniquely determine the set H:

$$H = H^{(a)} \cup H^{(b)} \cup (H^{(a)} \times H^{(b)}) \cup ((P_a \setminus H^{(a)}) \times (P_b \setminus H^{(b)})). \tag{30}$$

We have the following three cases (after possibly reversing the role of a and b):

(a) $H^{(a)} = C_{p_a}$ and $H^{(b)} = C_{p_b}$. Then

$$\begin{aligned} H &= \{x \oplus 0 | \langle p_a, x\rangle = 0\} \cup \{0 \oplus y | \langle p_b, y\rangle = 0\} \\ &\quad \cup \{x \oplus y | \langle p_a, x\rangle = \langle p_b, y\rangle = 0\} \cup \{x \oplus y | \langle p_a, x\rangle = \langle p_b, y\rangle = 1\} \\ &= \{x \oplus y | \langle p_a, x\rangle + \langle p_b, y\rangle = 0\} \\ &= \{x \oplus y | \langle p_a \oplus p_b, x \oplus y\rangle = 0\} \\ &= C_{p_a \oplus p_b}. \end{aligned} \tag{31}$$

(b) $H^{(a)} = H_{p_a}$ and $H^{(b)} = H_{p_b}$. Then

$$\begin{aligned} H &= \{x \oplus 0 | Q_{p_a}(x) = 0\} \cup \{0 \oplus y | Q_{p_b}(y) = 0\} \\ &\quad \cup \{x \oplus y | Q_{p_a}(x) = Q_{p_b}(y) = 0\} \cup \{x \oplus y | Q_{p_a}(x) = Q_{p_b}(y) = 1\} \\ &= \{x \oplus y | Q_{p_a}(x) + Q_{p_b}(y) = 0\} \\ &= \{x \oplus y | Q_{p_a \oplus p_b}(x \oplus y) = 0\} \\ &= H_{p_a \oplus p_b}. \end{aligned} \tag{32}$$

(c) $H^{(a)} = H_{p_a}$ and $H^{(b)} = C_{p_b}$. In this case, H would be

$$H = \{x \oplus y | Q_{p_a}(x) + \langle p_b, y\rangle = 0\}. \tag{33}$$

Since $a \geqslant 1$ and $b \geqslant 1$, we can pick from V_a and V_b two-dimensional symplectic subspaces W_1 and W_2 which are direct summands in the appropriate subspaces. Then

p_a (p_b) can be uniquely written as a sum of a vector p_1 (p_2) in W_1 (W_2) and one in its direct complement. Clearly, H intersects $W_1 \oplus W_2$ in

$$H \cap (W_1 \oplus W_2) = \{x_1 \oplus x_2 | Q_{p_1}(x_1) + \langle p_2, x_2\rangle = 0\}, \tag{34}$$

and this set should be a geometric hyperperplane in the $\mathcal{G}_2$, whose points are $P \cap (W_1 \oplus W_2)$. But denoting the three points in $P \cap W_i$ with a_i, b_i, c_i ($i \in \{1, 2\}$), one can write that

$$\langle a_1 \oplus a_2, b_1 \oplus b_2\rangle = \langle a_1, b_1\rangle + \langle a_2, b_2\rangle = 1 + 1 = 0 \tag{35}$$

and

$$(a_1 \oplus a_2) + (b_1 \oplus b_2) = (c_1 \oplus c_2), \tag{36}$$

so these points form a line in $\mathcal{G}_2$, and

$$\begin{aligned} &Q_{p_1}(a_1) + \langle p_2, a_2\rangle + Q_{p_1}(b_1) + \langle p_2, b_2\rangle + Q_{p_1}(c_1) + \langle p_2, c_2\rangle \\ &\quad = \langle a_1, b_1\rangle + \langle p_2, a_2 + b_2 + (a_2 + b_2)\rangle = 1 \end{aligned} \tag{37}$$

shows that an even number of points in this line belong to H which contradicts (H1). □

Given this algebraic characterization of points of different hyperplanes, it is easy to express the sum $A \boxplus B$ of any two geometric hyperplanes A and B.

Lemma 5. *Let $n \in \mathbb{N} + 1$, $a, b \in V_n$ and $\mathcal{G}_n = (P, L, \in)$. Then the following formulas hold:*

$$\begin{aligned} C_a \boxplus C_b &= C_{a+b} \\ H_a \boxplus H_b &= C_{a+b} \\ C_a \boxplus H_b &= H_{a+b}. \end{aligned} \tag{38}$$

Proof. Since all of the arising sets are defined as the zero locus of some $\mathbb{Z}_2$-valued function, we only have to observe that

$$\{x \in P | f_1(x) = 0\} \boxplus \{x \in P | f_2(x) = 0\} = \{x \in P | f_1(x) + f_2(x) = 0\} \tag{39}$$

holds.

Since $\langle a, x\rangle + \langle b, x\rangle = Q_a(x) + Q_b(x) = \langle a+b, x\rangle$ and $\langle a, x\rangle + Q_b(x) = Q_0(x) + \langle a, x\rangle + \langle b, x\rangle = Q_{a+b}(x)$, the statement follows. □

Now we are ready to prove that condition (V2) holds.

Theorem 3. *Let $n \in \mathbb{N} + 3$, and suppose that A, B, C are distinct geometric hyperplanes of $\mathcal{G}_n = (P, L, \in)$ such that $I = A \cap B \subseteq C$. Then $A \cap B = A \cap C$.*

Proof. For $C = A \boxplus B$, we have seen that $A \cap B = A \cap C = B \cap C$. We will show that there is no other possibility, i.e. $A \cap B \subseteq C$ implies $C \in \{A, B, A \boxplus B\}$.

By lemma 5 and the properties of $\boxplus$, we may assume that $A = C_a$ for some $a \in P$. We have two possibilities as follows.

(a) $B = C_b$ for some $b \in P$. Then $I = \{x \in P | \langle a, x\rangle = \langle b, x\rangle = 0\}$, so $I \cup \{0\} = (\mathrm{span}\{a, b\})^\perp$ is the symplectic complement of the subspace spanned by a and b. This means that $I \subseteq C_v = v^\perp \setminus \{0\}$ implies $v \in \mathrm{span}\{a, b\}$, or in other words, C_v is one of A, B and $A \boxplus B$.

In order to show that we cannot find a $v \in V_n$ such that $I \subseteq H_v$, pick two points $x, y \in I$ such that $\langle x, y\rangle = 1$. It is possible since $\dim V_n \geqslant 6$ when $n \geqslant 3$, $\dim \mathrm{span}\, I = \dim V_n - 2 > \frac{\dim V_n}{2}$ therefore span I cannot be isotropic. Then $x + y$ is also contained in I because $I \cup \{0\}$ is a linear subspace in V_n and $x \neq y$.

$$Q_v(x) + Q_v(y) + Q_v(x + y) = \langle x, y\rangle = 1 \tag{40}$$

shows that $\{x, y, x + y\}$ cannot be contained in H_v.

Table 1. Geometric hyperplanes in the incidence geometry associated to the n-qubit Pauli group.

General form	Number of points	Copies
C_p where $p \in V_n \setminus \{0\}$	$\frac{1}{2}4^n - 1$	$4^n - 1$
H_p where $Q_0(p) = 0$	$\frac{1}{2}(4^n + 2^n) - 1$	$\frac{1}{2}(4^n + 2^n)$
H_p where $Q_0(p) = 1$	$\frac{1}{2}(4^n - 2^n) - 1$	$\frac{1}{2}(4^n - 2^n)$

(b) $B = H_b$ for some $b \in V_n$. Then Q_b restricted to $a^\perp$ is a quadratic form of the maximal rank, and $I \cup \{0\}$ is its zero locus. Therefore, $\operatorname{span} I = a^\perp$, and it follows that if $I \subseteq C_v = v^\perp \setminus \{0\}$ then $v \in I^\perp = \operatorname{span}\{a\}$.

If for some $v \in V_n$ $I \subseteq H_v$, then

$$I = I \cap H_b \subseteq H_v \cap H_b \subseteq H_v \boxplus H_b = C_{v+b}, \tag{41}$$

which means that the only possibilities are $v = b$ and $v = a + b$. □

Remark. The statement is not true for $n = 2$. There the perp-sets of two commuting operators intersect in a single line which is obviously contained in a grid whose intersection with any of the two given perp-sets is a pentad [15, 16].

5. Automorphisms

Now that we have characterized all the geometric hyperplanes of $\mathcal{G}_n$, it is convenient to calculate how do automorphisms of $\mathcal{G}_n$ act on them. It is clear that every automorphism of V_n (i.e. a symplectic transformation) induces one of $\mathcal{G}_n$, and this group homomorphism is injective. Conversely, an automorphism of $\mathcal{G}_n$ respects the linear structure by lemma 5 and preserves the symplectic structure too, since it maps lines to lines. It follows then that $\operatorname{Aut}(\mathcal{G}_n) = Sp(2n, 2)$.

We have three types of geometric hyperplanes. One of them is C_p where $p \in V_n \setminus \{0\}$, and the two other types are of the form H_p, where $p \in V_n$. The type of this depends on the Arf invariant of Q_p which in turn equals $Q_0(p)$ which we will also call the Arf invariant of the hyperplane. The number of hyperplanes of each type is summarized in table 1.

Let $\mathcal{G}_n = (P, L, \in)$, where $P = V_n \setminus \{0\}$. The action of $Sp(2n, 2)$ on V_n induces an action on the Veldkamp space of $\mathcal{G}_n$. Since $Sp(2n, 2)$ is generated by symplectic transvections, we only have to calculate the action of these on the set of geometric hyperplanes. Let t_p denote the symplectic transvection determined by p:

$$t_p : V_n \to V_n; \qquad x \mapsto x + \langle p, x\rangle p. \tag{42}$$

A well-known fact is that the inverse of t_p is itself, that is, t_p is an involution.

Another important property is that if $\langle p, q\rangle = 0$ then t_p and t_q commute:

$$\begin{aligned} t_p t_q x &= t_p(x + \langle q, x\rangle q) \\ &= x + \langle q, x\rangle q + \langle p, x + \langle q, x\rangle q\rangle p \\ &= x + \langle q, x\rangle q + \langle p, x\rangle p + \langle q, x\rangle\langle p, q\rangle p. \end{aligned} \tag{43}$$

$\langle p, q\rangle = 0$ means that the last term is zero and the rest is symmetric in p and q.

Since t_p is linear, in particular, it fixes the zero vector, and acts as a permutation of P too. Being symplectic it permutes the elements of L too, and maps hyperplanes to hyperplanes. The action on these is given by the following.

Lemma 6. *Let* $\mathcal{G}_n = (P, L, \in)$ *and* $p \in V_n$. *Then*

$$\begin{aligned} t_p C_a &= C_{t_p a} \\ t_p H_a &= H_{a+(1+Q_a(p))p}. \end{aligned} \tag{44}$$

Proof.

$$\begin{aligned} t_p C_a &= \{t_p x | x \in V_n \setminus \{0\}, \langle a, x\rangle = 0\} \\ &= \{x \in P | \langle a, t_p x\rangle = 0\} \\ &= \{x \in P | \langle t_p a, x\rangle = 0\} \\ &= C_{t_p a} \end{aligned} \tag{45}$$

$$\begin{aligned} t_p H_a &= \{t_p x | x \in V_n \setminus \{0\}, Q_a(x) = 0\} \\ &= \{x \in P | Q_a(t_p x) = 0\} \\ &= \{x \in P | Q_a(x + \langle p, x\rangle p) = 0\} \\ &= \{x \in P | Q_0(x) + \langle p, x\rangle Q_0(p) + \langle x, \langle p, x\rangle p\rangle + \langle a, x + \langle p, x\rangle p\rangle = 0\} \\ &= \{x \in P | Q_0(x) + \langle Q_0(p)p + a + \langle a, p\rangle p + p, x\rangle = 0\} \\ &= H_{a+(1+Q_0(p)+\langle a,p\rangle)p}. \end{aligned} \tag{46}$$

□

Remark. In particular, t_p fixes C_a iff $\langle p, a\rangle = 0$ and fixes H_a iff $Q_a(p) = 1$.

It is well known that $Sp(2n, 2)$ acts transitively on the set of pairs of distinct nonzero vectors in V_n with a fixed symplectic product. This means that two hyperplanes of type C_p can be in two different positions relative to each other.

Our aim is to identify the possible relative positions of two geometric hyperplanes of type H_p. Clearly, the set

$$\{\{H_a, H_b\} | a, b \in V_n, a \neq b\} \tag{47}$$

splits to at least three invariant subsets under the action of $Sp(2n, 2)$, namely,

$$\{\{H_a, H_b\} | a, b \in V_n, a \neq b, Q_0(a) = Q_0(b) = 0\} \tag{48}$$

$$\{\{H_a, H_b\} | a, b \in V_n, a \neq b, Q_0(a) = Q_0(b) = 1\} \tag{49}$$

$$\{\{H_a, H_b\} | a, b \in V_n, Q_0(a) \neq Q_0(b)\}. \tag{50}$$

We will show that $Sp(2n, 2)$ acts transitively on each of these sets. This follows from the following lemma.

Lemma 7. *Let* $n \in \mathbb{N} + 3$, $\mathcal{G}_n = (P, L, \in)$ *and* $a, b, f \in V_n$ *three distinct vectors such that* $Q_0(a) = Q_0(b)$. *Then there exists an element in* $Sp(2n, 2)$ *fixing* H_f *and swapping* H_a *with* H_b.

Proof. There are two possibilities according to the value of $Q_f(a + b)$.

(a) If $Q_f(a + b) = 0$, then pick a point p in $C_{a+b} \cap H_a \setminus H_f$. This is possible since $C_{a+b} \boxplus H_a = H_b \neq H_f$ and (V2) holds. Now let $q = a + b + p$. Then since H_f is a geometric hyperplane, $a + b \in H_f$ and $p \notin H_f$, it follows that the line $\{q, p, a + b\}$ intersects H_f in $a + b$ and $Q_f(q) = 1$.

Also, we have that

$$\begin{aligned} Q_b(a+b+p) &= Q_0(a+b+p) + \langle b, a+b+p\rangle \\ &= Q_0(a) + Q_0(b) + Q_0(p) + \langle a, p\rangle \\ &= Q_a(p) = 0. \end{aligned} \tag{51}$$

It is clear then by equation (44) that both t_p and t_q fix H_f, $t_p H_a = H_{a+p}$ and $t_q H_b = H_{b+q} = H_{a+p}$. It follows that $t_q t_p H_a = H_b$. Since $\langle p, q\rangle = 0$, t_p and t_q are two commuting involutions, which implies that $t_q t_p$ itself is an involution, and it swaps H_a and H_b.

(b) If $Q_f(a+b) = 1$ then by equation (44), t_{a+b} fixes H_f and

$$Q_a(a+b) = Q_0(a) + Q_0(b) + \langle a, b\rangle + \langle a, a+b\rangle = 0 \tag{52}$$

implies that $t_{a+b} H_a = H_b$. □

Remark. This means also that $Sp(2n, 2)$ acts 2-transitively (transitively on pairs) on its two orbits of geometric hyperplanes of type H.

6. Veldkamp lines

Our considerations in the previous section show that there are two types of Veldkamp lines incident with three C-hyperplanes and three types of lines which are incident with one C-hyperplane and two H-hyperplanes. In this section we study the structure of these lines, i.e. the pairwise intersections of geometric hyperplanes.

By lemma 7 we only have to consider five special cases. The first type of line in the Veldkamp space of $\mathcal{G}_n$ we study is the one connecting C_a and C_b, where $\langle a, b\rangle = 0$. Their intersection contains points of the symplectic complement of span$\{a, b\}$. Since this subspace is isotropic, the symplectic complement is isomorphic to $V_{n-2} \oplus W_1$, where W_1 is a two-dimensional vector space with an identically 0 bilinear form. Therefore, the intersection has $4^{n-1} - 1$ points. The number of lines of this type is the same as the number of lines in $\mathcal{G}_n$.

Similarly, when $\langle a, b\rangle = 1$, then the intersection of C_a and C_b is also the symplectic complement of span$\{a, b\}$ minus the zero vector. But in this case the symplectic form which is restricted to the complement is nondegenerate, meaning that the intersection as an incidence geometry is isomorphic to $\mathcal{G}_{n-1}$ which has $4^{n-1} - 1$ points. Each two-dimensional symplectic subspace in V_n gives rise to one such line; hence, the number of them is

$$\frac{(4^n - 1)\cdot 4^{n-1}}{3}. \tag{53}$$

The next case is a line connecting two H-hyperplanes with Arf invariants equal to 0. By lemma 7, we can choose any two such hyperplanes. For simplicity, we will work with H_0 and H_a, where $a = I \ldots IX = (0, 0, \ldots, 0, 1)$. Now $x \in H_0 \cap H_a$ implies that $x \in C_a$, so x is of the form

$$x = (x_1, x_2, \ldots, x_{2n-2}, 0, x_{2n}). \tag{54}$$

The value of $Q_0(x)$ is independent of x_{2n} and equals

$$Q_0((x_1, x_2, \ldots, x_{2n-2})) = x_1x_2 + \cdots + x_{2n-3}x_{2n-2}, \tag{55}$$

where Q_0 also denotes a quadratic form on V_{n-1}, but this should not be a source of confusion. As we are free to choose the value of x_{2n}, it follows that the intersection contains

$$2\left[\tfrac{1}{2}(4^{n-1} + 2^{n-1})\right] - 1 = 4^{n-1} + 2^{n-1} - 1 \tag{56}$$

points where -1 is for excluding the zero vector. We have

$$\begin{aligned}\binom{\frac{4^n+2^n}{2}}{2} &= \frac{1}{8}(4^n+2^n)(4^n+2^n-2)\\ &= 2^{n-3}(2^n+1)(2^n-1)(2^n+2)\\ &= 2^{n-3}(4^n-1)(2^n+2)\end{aligned} \tag{57}$$

lines of this type.

For the line type connecting two H-hyperplanes with different Arf invariants, we choose H_0 and H_a, $a = I \ldots IY = (0, 0, \ldots, 0, 1, 1)$ as the representative. Now if $x \in H_0 \cap H_a$ then $x \in C_a$, so $x = (x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}, x_{2n-1})$. This implies that $Q_0(x)$ equals

$$Q_0((x_1, x_2, \ldots, x_{2n-2})) + x_{2n-1}, \tag{58}$$

meaning that we can choose the first $2n - 2$ coordinates freely, and this uniquely determines x_{2n-1}. Moreover, since the last two coordinates do not contribute to the value of the symplectic form of two such vectors, it follows that the intersection is again isomorphic to $\mathcal{G}_{n-1}$ as an incidence geometry. But this time it is embedded differently into $\mathcal{G}_n$ since the span of its points is $(2n - 1)$-dimensional unlike the symplectic complement of a symplectic two-dimensional subspace whose span is $2n - 2$ dimensional.

A pair of H-hyperplanes with different Arf invariants can be chosen in

$$\begin{aligned}\tfrac{1}{2}(4^n+2^n)\cdot\tfrac{1}{2}(4^n-2^n) &= \tfrac{1}{4}(2^n)^2(2^n+1)(2^n-1)\\ &= 4^{n-1}(4^n-1)\end{aligned} \tag{59}$$

ways.

The last case is a line containing two H-hyperplanes with Arf invariant 1. Our choice is $a = I \ldots IY = (0, 0, \ldots, 0, 1, 1)$ and $b = I \ldots IXY = (0, \ldots, 0, 1, 1, 1)$, and the two hyperplanes are H_a and H_b. Then for $x \in H_a \cap H_b$, we have that $x \in C_{a+b}$ meaning that $x = (x_1, x_2, \ldots, x_{2n-4}, 0, x_{2n-2}, x_{2n-1}, x_{2n})$ Now Q_a equals

$$Q_0((x_1, x_2, \ldots, x_{2n-4})) + x_{2n-1} + x_{2n} + x_{2n-1}x_{2n}, \tag{60}$$

which does not depend on x_{2n-2}. If any of the last two coordinates is 1, then the sum of the last three terms is 1, so we have $\frac{1}{2}(4^{n-2} - 2^{n-2})$ possibilities for the values of the first $n - 4$ coordinates. On the other hand, if both x_{2n-1} and x_{2n} are 0, then we have $\frac{1}{2}(4^{n-2} + 2^{n-2})$ choices for the first $n - 4$ coordinates. In any case, we are free to choose the value of x_{2n-2} except for the case of $x = 0$, so the intersection has

$$2\left(3\tfrac{1}{2}(4^{n-2}-2^{n-2}) + \tfrac{1}{2}(4^{n-2}+2^{n-2})\right) - 1 = 4^{n-1} - 2^{n-1} - 1 \tag{61}$$

points. The number of lines of this type is

$$\begin{aligned}\binom{\frac{4^n-2^n}{2}}{2} &= \frac{1}{8}(4^n-2^n)(4^n-2^n-2)\\ &= 2^{n-3}(2^n-1)(2^n+1)(2^n-2)\\ &= 2^{n-3}(4^n-1)(2^n-2).\end{aligned} \tag{62}$$

The above results are summarized in table 2.

7. Special cases

After the discussion of general properties of the incidence geometries of the n-qubit Pauli group, we turn to some special cases. As was already mentioned, $\mathcal{G}_1$ consists of three points

Table 2. Veldkamp lines in the incidence geometry associated to the n-qubit Pauli group.

	Hyperplanes of type			
C_a	H_a, $Q_0(a) = 0$	H_a, $Q_0(a) = 1$	Intersection size	Number of copies
3	0	0	$4^{n-1} - 1$	$\frac{1}{3}(4^n - 1)(4^{n-1} - 1)$
3	0	0	$4^{n-1} - 1$	$\frac{1}{3}4^{n-1}(4^n - 1)$
1	2	0	$4^{n-1} + 2^{n-1} - 1$	$2^{n-3}(4^n - 1)(2^n + 2)$
1	1	1	$4^{n-1} - 1$	$4^{n-1}(4^n - 1)$
1	0	2	$4^{n-1} - 2^{n-1} - 1$	$2^{n-3}(4^n - 1)(2^n - 2)$

and no lines. This is not very interesting, as all proper subsets arise as geometric hyperplanes and intersections of those are all subsets with at most one point.

The $n = 2$ case is rather peculiar as $\mathcal{G}_2$ is the unique generalized quadrangle of order 2 having 15 points and 15 lines. It was already studied in detail in [16, 17]. Our present results can be regarded as a generalization of this case. It is interesting to note that in this case an H-hyperplane with Arf invariant 1 consists of five points reaching the lower bound of lemma 1. These do not have any lines, and hence are ovoids, corresponding to sets of mutually anticommuting Pauli operators.

The H-hyperplanes with Arf invariant 0 contain nine points and six lines forming a subquadrangle $GQ(2, 1)$ also known as a grid. On the quantum information theoretic side these are Mermin squares which are used for a simplified proof of the Kochen–Specker theorem [18].

For $n = 3$, the incidence geometry $\mathcal{G}_n$ has 63 points and 315 lines. The H-hyperplanes with Arf invariant 1 here have 27 points and 45 lines. These hyperplanes as incidence geometries are isomorphic to the generalized quadrangle $GQ(2, 4)$. Fixing one of them, all of its geometric hyperplanes can be obtained by intersecting it with every other hyperplane in $\mathcal{G}_3$. They are $GQ(2, 2)$-s and perp-sets containing 11 points [19]. The importance of this object was already known in the context of the black hole analogy [8]; the novelty here is the natural description in terms of three-qubit operators.

It is also interesting that keeping a certain set of 63 lines of $\mathcal{G}_3$ one can obtain the split Cayley hexagon of order 2 [7]. Since keeping all points and deleting lines weaken the condition of being a geometric hyperplane, all hyperplanes of $\mathcal{G}_3$ can also be viewed as hyperplanes of the hexagon, but the latter contains many more types of hyperplanes [20]. From the physical point of view, there are some hints that the Cayley hexagon might have a role in understanding the connection between the three-qubit Pauli group and the $E_{7(7)}$-symmetric entropy formula of black holes in $N = 8$ $D = 4$ supergravity.

We also have two special geometric hyperplanes in $\mathcal{G}_n$ for any n if we fix the representation of the n-qubit Pauli group as tensor products of the usual Pauli matrices. These are H_0 and $H_{YY\ldots Y}$. As was already mentioned, the first one consists of Pauli operators with symmetric matrices as representatives. The latter one contains the operators built from an even number of nontrivial (i.e. X, Y, or Z) matrices. These n-qubit operators are exactly the self-dual ones with respect to the Wootters spin-flip [21] transformation (apart from the identity matrix).

For $n = 3$, these Wootters self-dual Pauli operators form an H-hyperplane giving rise to a $GQ(2, 4)$ underlying the geometry of the $E_{6(6)}$ symmetric entropy formula for black holes and black strings. Of course, this hyperplane is just one from the aforementioned 28 possible ones with Arf invariant equals 1. They also have the structure of a $GQ(2, 4)$. Clearly, all of these hyperplanes can be used to describe the same $E_{6(6)}$ symmetric black hole and black string entropy formula, but with the points having different noncommutative labelings. This situation can be regarded as the finite geometric analog of the standard usage of different local coordinates for the underlying manifold in (pseudo)Riemannian geometry. As also emphasized in our recent paper [8], the important novelty here is the intrinsically noncommutative nature of these coordinates. Using the unified framework as developed in this paper the mathematical and physical implications of these observations are certainly worth exploring further.

8. Conclusion

We have associated a point-line incidence geometry to every n-qubit-generalized Pauli group, from which the group can fully be recovered. This contains points and lines of a symplectic polar space of rank n and order 2 which describes the commutation relations of the Pauli group [4].

For $n \geqslant 3$, this incidence structure has a Veldkamp space in the stronger sense which enables us to identify distinguished subsets of the group independently from its representation. These are the geometric hyperplanes (Veldkamp points) and the intersections of pairs of hyperplanes (Veldkamp lines).

This formalism also creates a nice unifying picture of finite geometric results in connection with the black hole analogy. Namely, both the generalized quadrangle with (2, 4) parameters, which is intimately connected to the $E_{6(6)}$-symmetric black hole entropy formula [8], and the split Cayley hexagon of order 2, which is related to the $E_{7(7)}$-symmetric black hole entropy formula in $N = 8$ $D = 4$ supergravity [7], can be found as a subgeometry in the incidence structure describing the Pauli group of three qubits.

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